Homework 3
Geometry

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Proposition 0.1 (Exercise 2-3b). *The antipodal map $\alpha : S^n \to S^n$ given by $x \mapsto -x$ is smooth.*

Proof. We need to show that for every $p \in S^n$, there exist smooth charts (U, ϕ) and (V, ψ) with $p \in U$, $\alpha(p) \in V$, $\alpha(U) \subset V$, and $\psi \circ \alpha \circ \phi^{-1} : \phi(U) \to \psi(V)$ smooth.

Let $x = (x^1, \dots x^{n+1}) \in S^n \subset \mathbb{R}^{n+1}$. Then $x^i \neq 0$ for some i, so $x \in U_i^+$ or $x \in U_i^-$ (where $U_i^+ = \{x \in S^n : x^i > 0\}$ and U_i^- is analogous). Without loss of generality, suppose $x \in U_i^+$. Then $\alpha(x) = -x \in U_i^-$, and $\alpha(U_i^+) = U_i^-$, so in particular the inclusion $\alpha(U_i^+) \subset U_i^-$ is satisfied. Finally, we compute the composition $\phi_i^\pm \circ \alpha \circ (\phi_i^\pm)^{-1} : \phi_i^\pm(U_i^+) \to \phi_i^\pm(U_i^-)$. Let $u = (u^1, \dots u^n) \in \phi_i^\pm(U_i^+)$.

$$\begin{aligned}
\phi_i^\pm \circ \alpha \circ (\phi_i^\pm)^{-1}(u^1, \dots u^n) &= \phi_i^\pm \circ \alpha(u^1, \dots u^{i-1}, \sqrt{1 - |u|^2}, u^i, \dots u^n) \\
&= \phi_i^\pm(-u^1, \dots - u^{i-1}, -\sqrt{1 - |u|^2} - u^i, \dots, -u^n) \\
&= (-u^1, \dots, -u^{i-1}, -u^i, \dots, -u^n) \\
&= -u
\end{aligned}$$

and thus this composition is smooth, hence α is smooth. □

Proposition 0.2 (Exercise 2-6). *Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be smooth, and suppose that there exists $d \in \mathbb{Z}$ such that $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}, x \in \mathbb{R}^{n+1} \setminus \{0\}$. Then $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ given by $[x] \mapsto [P(x)]$ is well-defined and smooth.*

Proof. To show that $\tilde{P}$ is well-defined, we need to show that $[x] = [y] \implies \tilde{P}[x] = \tilde{P}[y]$. Suppose $[x] = [y]$. Then $y = \lambda x$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then

$$\tilde{P}[y] = [P(y)] = [P(\lambda x)] = [\lambda^d P(x)] = [P(x)] = \tilde{P}[x]$$

thus $\tilde{P}$ is well-defined. Now we show that $\tilde{P}$ is smooth.

Let $[x] \in \mathbb{RP}^n$ and fix $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Let $\pi_n : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the standard projection $x \mapsto [x]$. Then set

$$\begin{aligned}
\tilde{U}_i &= \{y \in \mathbb{R}^{n+1} \setminus \{0\} | y^i \neq 0\} \\
U_i &= \pi_n(\tilde{U}_i)
\end{aligned}$$

Since $x \in \mathbb{R}^{n+1} \setminus \{0\}$, we have $x^i \neq 0$ for some i, so $x \in \tilde{U}_i$ and $[x] \in U_i$. Since $\tilde{U}_i$ is a saturated open set, U_i is an open neighborhood of $[x]$. Set

$$\tilde{V}_i = P(\tilde{U}_i) = \{y \in \mathbb{R}^{n+1} \setminus \{0\} | y^i \neq 0\}$$

The second equality follows because P maps lines to lines. Then we set $V_i = \pi_k(\tilde{V}_i)$, and we have

$$V_i = \pi_k(\tilde{V}_i) = \pi_k \circ P(\tilde{U}_i) = \tilde{P} \circ \pi_n(\tilde{U}_i) = \tilde{P}(U_i)$$

Then $\tilde{P}([x]) \in V_i$ and $\tilde{P}(U_i) \subset V_i$, and V_i is open because $\tilde{V}_i$ is a saturated open set. Define $\phi_i : U_i \to \mathbb{R}^n, \psi_i : V_i \to \mathbb{R}^n$ as in Example 1.5, as

$$\begin{aligned} \phi_i[x^1, \ldots, x^{n+1}] &= \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i} \right) \\ \psi_i[x^1, \ldots, x^{k+1}] &= \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{k+1}}{x^i} \right) \end{aligned}$$

We claim that $\psi_i \circ \tilde{P} \circ \phi_i^{-1} : \psi_i(V_i) \to \phi_i(U_i)$ is smooth. We compute

$$\begin{aligned} \psi_i \circ \tilde{P} \circ \phi_i^{-1}(u^1, \ldots, u^n) &= \psi_i \circ \tilde{P}([u^1, \ldots, u^{i-1}, 1, u^i, \ldots u^n]) \\ &= \psi_i \circ \tilde{P} \circ \pi_n(u^1, \ldots, u^{i-1}, 1, u^i, \ldots u^n) \\ &= \psi_i \circ \pi_k \circ P(u^1, \ldots, u^{i-1}, 1, u^i, \ldots u^n) \end{aligned}$$

Since ψ_i, π_k, and P are smooth, their composition is smooth.

Thus we have shown that for every $[x] \in \mathbb{RP}^n$, we have smooth charts $(U_i, \phi_i), (V_i, \psi_i)$ with $[x] \in U_i$, $\tilde{P}([x]) \in V_i$, $\tilde{P}(U_i) \subset V_i$, and $\psi_i \circ \tilde{P} \circ \phi_i^{-1} : \phi_i(U_i) \to \psi_i(V_i)$ smooth. Hence $\tilde{P}$ is smooth.

As an assistance in keeping track of the argument, we provide the following commutative diagram:

$$\begin{array}{ccccccc} \tilde{U}_i & \hookrightarrow & \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{P} & \mathbb{R}^{k+1} \setminus \{0\} & \hookleftarrow & \tilde{V}_i \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_k} \\ U_i & \hookrightarrow & \mathbb{RP}^n & \xrightarrow{\tilde{P}} & \mathbb{RP}^k & \hookleftarrow & V_i \\ \downarrow{\scriptstyle \phi_i} & & & & & & \downarrow{\scriptstyle \psi_i} \\ \phi_i(U_i) & \hookrightarrow & \mathbb{R}^n & & \mathbb{R}^k & \hookleftarrow & \psi_i(V_i) \end{array}$$

□

Lemma 0.3 (for Exercise 2-7)**.** *Let M be a nonempty smooth n-manifold with or without boundary where $n \geq 1$. If $f_1, \ldots f_k \in C^\infty(M)$ such that* $\operatorname{supp}(f_i) \neq \emptyset$ *and* $i \neq j \implies \operatorname{supp}(f_i) \cap \operatorname{supp}(f_j) = \emptyset$*, then $f_1, \ldots f_k$ are linearly independent.*

Proof. Let $f_1, \dots f_k$ be as described, and suppose that

$$\sum_{i=1}^{k} a^i f_i = 0$$

for some real scalars $a^1, \dots a^k$. For each α, $\operatorname{supp}(f_\alpha) \neq \emptyset$, so there exists $x_\alpha \in \operatorname{supp}(f_\alpha)$ with $f_\alpha(x_\alpha) \neq 0$. (If no such x_α exists, then $\operatorname{supp}(f_\alpha)$ is the closure of the empty set, which is empty.) Then since the supports are disjoint, $x_\alpha \notin \operatorname{supp}(f_j)$ for $j \neq \alpha$, so $f_j(\alpha) = 0$ (for $j \neq \alpha$). Then

$$0 = \left(\sum_{i=1}^{k} a^i f_i \right)(x_\alpha) = \sum_{i=1}^{k} a^i f_i(x_\alpha) = a^\alpha f_\alpha(x_\alpha)$$

Since $f_\alpha(x_\alpha) \neq 0$, this implies that $a^\alpha = 0$. Since α was arbitrary, this means that $a^1 = \dots = a^k = 0$, hence $f_1, \dots f_k$ are linearly independent. □

Proposition 0.4 (Exercise 2-7)**.** *Let M be a nonempty smooth n-manifold with or without boundary where $n \geq 1$. Then $C^\infty(M)$ is infinite dimensional.*

Proof. Suppose $C^\infty(M)$ is finite dimensional with dimension k. M is nonempty, so there exists $p \in M$, and then there must exist a chart (U, ϕ) with $p \in U$ and $U \cong B(0, 1)$. So U is uncountable. Choose distinct points $x_1, \dots x_{k+1} \in U$. Because M is Hausdorff, for each pair x_i, x_j with $i \neq j$ there exist open sets U_i^j, U_j^i with $x_i \in U_i, x_j \in U_j$, and $U_i^j \cap U_j^i = \emptyset$. Let

$$U_i = \cap_{j \neq i} U_i^j$$

Each U_i is a finite intersection of open sets, so U_i is open. Since $x_i \in U_i^j$ for each j, we have $x_i \in U_i$. Since $x_j \notin U_i^j$ for each j, $x_j \notin U_i$ unless $i = j$. Thus $U_1, \dots U_{k+1}$ are pairwise disjoint open neighborhoods of $x_1, \dots x_{k+1}$.

By Proposition 2.25, for each i there exists a smooth bump function f_i for $\{x_i\}$ supported on U_i. Thus we have smooth functions $f_1, \dots f_{k+1}$ with nonempty disjoint supports, so by the previous lemma these are linearly independent. This contradicts the assumption that $C^\infty(M)$ has dimension k. Since k was arbitrary, this means that $C^\infty(M)$ cannot have any finite dimension, so it is infinite dimensional. □

Proposition 0.5 (Exercise 2-10a)**.** *Let M, N be manifolds with $C(M), C(N)$ being the respective algebras of continuous functions $M \to \mathbb{R}, N \to \mathbb{R}$. Let $F : M \to N$ be continuous, and define $F^* : C(N) \to C(M)$ by $f \mapsto f \circ F$. Then the map F^* is linear.*

Proof. Let $a \in \mathbb{R}$, and $f, g \in C(N)$. Then

$$\begin{aligned} F^*(af + g)(x) &= (af + g) \circ F(x) \\ &= (af + g)(F(x)) \\ &= af(F(x)) + g(F(x)) \\ &= a(f \circ F)(x) + (g \circ F)(x) \\ &= a(F^*(f))(x) + F^*(g)(x) \end{aligned}$$

Thus $F^*(af + g) = aF^*(f) + F^*(g)$, so F^* is linear. □

Lemma 0.6 (Exercise 2.3, needed for Exercise 2-10b)**.** *Let M be a smooth n-manifold, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, ϕ) for M.*

Proof. Let (U, ϕ) be a smooth chart for M, and let $p \in U$. Since f is smooth, there exists a chart (V_p, ψ_p) such that $p \in V_p$ and $f \circ \psi_p^{-1} : \psi_p(V_p) \to \mathbb{R}^n$ is smooth. Since $(U, \phi), (V_p, \psi_p)$ are smoothly compatible, the transition map $\psi_p \circ \phi^{-1} : \phi(U \cap V_p) \to \psi_p(U \cap V_p)$ is a diffeomorphism, that is, it is smooth. Then

$$(f \circ \psi_p^{-1}) \circ (\psi_p \circ \phi^{-1}) = f \circ \phi^{-1} : \phi(U \cap V_p) \to \mathbb{R}^k$$

Thus for every $\phi(p) \in \phi(U)$, there is a neighborhood $\phi(U \cap V_p)$ such that the restriction of $f \circ \phi^{-1}$ to this neighborhood is smooth. Hence $f \circ \phi^{-1}$ is smooth (using Proposition 2.6a). □

Proposition 0.7 (Exercise 2-10b)**.** *Let M, N be smooth manifolds and $F : M \to N$. Let $F^* : C(N) \to C(M)$ be the induced map $f \mapsto f \circ F$. Then F is smooth iff $F^*(C^\infty(N)) \subset C^\infty(M)$.*

Proof. First suppose that F is smooth. Then for $f \in C^\infty(N)$, f is smooth so $F^*(f) = f \circ F$ is a composition of smooth functions, so it is smooth. That is, $F^*(f) \in C^\infty(M)$.

Now suppose that $F^*(C^\infty(N)) \subset C^\infty(M)$. Let $x \in M$, and let $(\tilde{U}, \tilde{\phi})$ be a smooth chart with $x \in \tilde{U} \subset M$. We also have $F(x) \in N$, so let (V, ψ) be a smooth chart with $F(x) \subset V \subset N$. Then let $U = F^{-1}(V) \cap \tilde{U}$ and $\phi = \tilde{\phi}|_U$, so now we have a new chart (U, ϕ) with $F(U) \subset V$. Let n be the dimension of N, and let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the projection $x \mapsto x^i$. Then $\pi_i \circ \psi : V \to \mathbb{R}$ is smooth, so by hypothesis,

$$F^*(\pi_i \circ \psi) = \pi_i \circ \psi \circ F$$

is smooth for each i. Then by the above lemma (Exercise 2.3),

$$\pi_i \circ \psi \circ F \circ \phi^{-1} : \phi(U) \to \mathbb{R}$$

is smooth for each i, so $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth. Hence F is smooth (for every $x \in M$, we have a charts $(U, \phi), (V, \psi)$ with $x \in U$, $F(U) \subset V$, and $\psi \circ F \circ \phi^{-1}$ smooth). □

Proposition 0.8 (Exercise 2-10c)**.** *Let M, N be smooth manifolds and let $F : M \to N$ be a homeomorphism. Define $F^* : C(N) \to C(M)$ by $f \mapsto f \circ F$. Then F is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.*

Proof. Suppose that F is a diffeomorphism. Since F is smooth, by part (b) we have $F^*(C^\infty(N)) \subset C^\infty(M)$. Since F^{-1} exists and is smooth, by part (b) we have $(F^{-1})^*(\mathbb{C}^\infty(M)) \subset C^\infty(N)$. Additionally,

$$((F^{-1})^* \circ F^*)(f) = (F^{-1})^*(f \circ F) = f \circ F \circ F^{-1} = f$$
$$(F^* \circ (F^{-1})^*)(f) = F^*(f \circ F^{-1}) = f \circ F^{-1} \circ F = f$$

so $(F^{-1})^* = (F^*)^{-1}$. Now we need to show that $F^*, (F*)^{-1}$ are homomorphisms. Let $f, g \in C^\infty(N)$ and $h, k \in C^\infty(M)$.

$$\begin{aligned} F^*(fg) &= (fg) \circ F = (f \circ F)(g \circ F) = F^*(f)F^*(g) \\ (F^*)^{-1}(hk) &= (F^{-1})^*(hk) = (hk) \circ F^{-1} = (h \circ F^{-1})(k \circ F^{-1}) \\ &= (F^{-1})^*(h)(F^{-1})^*(k) = (F^*)^{-1}(h)(F^*)^{-1}(k) \end{aligned}$$

Thus $F^*, (F^*)^{-1}$ are homomorphisms that are inverse to each other, thus both are isomorphisms.

Now suppose that F^* restricts to an isomorphism on from $C^\infty(N)$ to $C^\infty(M)$. By hypothesis, F is a homeomorphism, and by part (b), F is smooth, so to show that F is a diffeomorphism we just need to show that F^{-1} is smooth. As shown above, $(F^{-1})^* = (F^*)^{-1}$, and since $F^*(C\infty(N)) \subset C\infty(M)$, we have $(F^{-1})^*(C^\infty(M)) \subset C^\infty(N)$. Thus by part (b), F^{-1} is smooth. Hence F is a diffeomorphism. □

Proposition 0.9 (Exercise 2-14)**.** *Let M be a smooth manifold with disjoint closed subsets A, B. There exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$ and $f^{-1}(0) = A$ and $f^{-1}(1) = B$.*

Proof. By Theorem 2.29, there exist smooth, nonnegative functions f_A, f_B such that $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$. Let $f : M \to \mathbb{R}$ be defined by

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$$

Note that the denominator is never zero, since A, B are disjoint, and f_A only vanishes on A and f_B only vanishes on B (and both are nonnegative). Thus this function is smooth, and nonnegative on M. It is also bounded by $0, 1$ as $f_A \leq f_A + f_B$. In particular, for $a \in A, b \in B$,

$$\begin{aligned} f(a) &= \frac{f_A(a)}{f_A(a) + f_B(a)} = \frac{0}{0 + f_B(a)} = 0 \\ f(b) &= \frac{f_A(b)}{f_A(b) + f_B(b)} = \frac{f_A(b)}{f_A(b) + 0} = 1 \end{aligned}$$

Note that for $x \notin A \cup B$, $f_A(x) \neq 0$ so $f(x) \neq 0$, and $f_B(x) \neq 0$ so $f(x) \neq 1$. Thus f is the required function. □